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An Approach for Estimating Vibration Characteristics of Nonuniform Rotor Blades

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A method is presented for determining the vibration characteristics of a rotating blade whose cross-sectional dimensions or mechanical properties may vary sharply or even discontinuously along its length. The coupled flapwise bending, chordwise bending, and torsional vibration of the blade is analyzed by the method of the new quotient which is based on a variational statement proposed by Nemat-Nasser. In this approach, the nonuniform blade properties may be approximated by step (piecewise constant) functions. Two illustrative examples are given, and the results are compared with available experimental data and other numerical solutions. The comparison shows that the method of the new quotient yields very good results.

Nomenclature

B_1, B_2 = section constants, Eqs. (2b) and (2c)
E = Young's modulus
e = distance between mass and elastic axes, positive when mass axis lies ahead of elastic center
e_A = distance beween area centroid of tensile member and elastic axis, positive for centroid forward
e_0 = distance at root between elastic axis and axis about which blade is rotating, positive when elastic axis lies ahead
G = shear modulus
I_1, I_2 = bending moments of inertia about major and minor neutral axes, respectively (both pass through centroid of cross-sectional area effective in carrying tension)
J = torsional stiffness
k_A = polar radius of gyration of cross-sectional area effective in carrying tensile stresses about elastic axis
k_m = polar radius of gyration of cross-sectional mass about elastic axis ($k_m^2 = k_{m1}^2 + k_{m2}^2$)
k_{m1}, k_{m2} = mass radii of gyration about major neutral axis and about an axis perpendicular to chord through the elastic axis, respectively
M_y, M_z = resultant moments in the y and z directions, respectively
m = mass of beam per unit length
Q = resultant cross-sectional torque about elastic axis
R = blade length
T = tension in blade, Eq. (2a)
t = thickness of cross section
V_y, V_z = shear forces in the y and z directions, respectively
w = amplitude of simple harmonic lateral displacement normal to the plane of rotation
v = amplitude of simple harmonic lateral displacement in the plane of rotation
x,y,z = right-handed Cartesian coordinate system which rotates with blade in such a manner that the x axis falls along initial or undeformed position of elastic axis; the y axis positive toward leading edge, the z axis positive upward
x_0 = distance from the axis of rotation to the root of the elastic blade
β = blade angle of station x prior to any deformation, positive when leading edge is upward
η = cross-sectional axis along major neutral axis
η_{te}, η_{le} = values of η for trailing edge and leading edge, respectively
ϕ = amplitude of simple harmonic torsional deformation, positive leading edge upward
Ω = angular velocity of rotation
ω = frequency of vibration
$(\ )'$ = $d(\ )/dx$

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I. Introduction

THE determination of the dynamic characteristics of rotating blades is of fundamental importance in the design of many engineering components, including turbine blades, compressor blades, aircraft propeller blades, and helicopter rotor blades. It is necessary that the natural frequencies be determined accurately in order to obtain a blade design that is as free from resonance as possible. The corresponding natural modes may be combined to obtain series solutions for transient response problems, and also are of great importance in the flutter analysis.

Based on a linear theory for the deformation of a rotating blade, Houbolt and Brooks[1] derived the general differential equations of motion for combined flapwise bending, chordwise bending, and torsion of a pretwisted nonuniform blade with asymmetrical cross section, and Carnegie[2] established the governing equations to analyze the coupled bending-bending-torsion vibration of a cantilever blade mounted on the periphery of a rotating disk. The results in Ref. 2 may be used to analyze blades with fixed-free ends, and those in Ref. 1 are applicable to blades with various end conditions. The differential equations of motion for the rotor blades are, in general, too complicated to be solved exactly. Various methods have been developed to obtain approximate solutions of either the general governing equations or the special cases; for example, we mention here the integrating matrix method,[3-5] the transfer matrix method,[6,7] the Runge-Kutta method,[8,9] the finite-difference method,[10,11] the finite-element method,[12,13] and the Galerkin method.[1,14]

The energy approach is also commonly used for the vibration analysis of blades. Carnegie,[2] DiPrima and Handelman,[15] and Dawson[16] have employed the Rayleigh quotient with the Rayleigh-Ritz procedure to determine the vibration characteristics of blades with uniform cross sections for various cases. Yntema[17] used the same method to analyze the flapwise bending vibration of nonuniform blades; in this study, the variable or discontinuous mass and stiffness distributions were approximated by linear relations. Houbolt and Brooks[1] made use of the Rayleigh energy method to estimate the natural frequencies of pretwisted blades with nonuniform asymmetrical cross sections.

The Rayleigh energy method is very effective for elastic systems with continuous elastic properties and mass distributions. However, if the blade has discontinuously varying properties along the spanwise direction, the Rayleigh energy method gives bad estimates for the eigenfrequencies, and the corresponding estimates for the moment mode shapes near the points of discontinuity are generally very poor. This is because the approximating function for the deflection (twisting angle) cannot satisfy the condition of continuity of the bending (twisting) moment along the blade length.

Lang and Nemat-Nasser[18] have introduced the method of the new quotient to analyze the vibration and buckling of nonrotating beams with discontinuously varying dimension; in Ref. 18 the advantage of the new approach over the Rayleigh energy method has been demonstrated by several numerical examples. The method of the new quotient, proposed and illustrated in Refs. 19-21, is based on a mixed variational principle that allows independent variations of various field quantities; e.g., the deflection, the twisting angle, and the moments. It has the advantage that the continuity conditions for the moments are naturally built into the calculation scheme. Usually the computational effort required for this method is almost the same as that for the Rayleigh energy method, but the results are more accurate.

In the present work, the method of the new quotient is used to analyze the vibration characteristics of nonuniform rotor blades. The governing equations for the coupled bending-bending-torsion vibration of pretwisted nonuniform blades are taken from the work of Houbolt and Brooks[1] (see Sec. II). The corresponding new quotient is formulated in Sec. III. Then, the Rayleigh-Ritz procedure is employed to estimate the eigenfrequencies and the eigenmode shapes. Illustrative examples are presented in Sec. IV. The blade properties, which are selected from other papers, are expressed in the form of either experimental data or discontinuous functions.

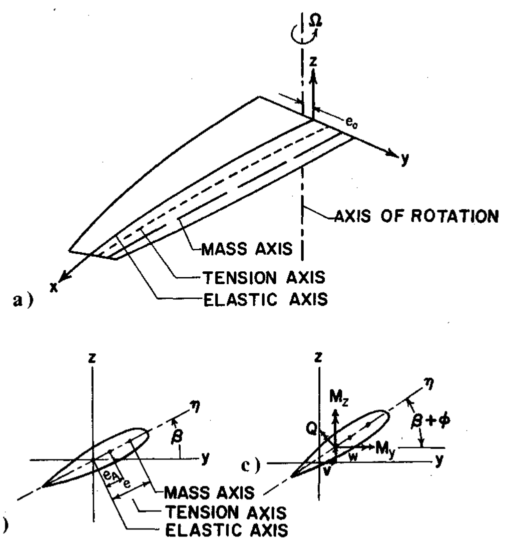

Fig. 1 Blade treated in Eq. (1): a) Configuration of the twisted rotating blade; b) Coordinates (undeformed position); c) Displacements and moments.

In Sec. V, the numerical results obtained by the method of the new quotient are compared with the available experimental data and other numerical solutions. This comparison shows that the method yields very good results, and that estimates for higher modes can be obtained without any difficulties.

II. Basic Equations

The general differential equations of motion for combined flapwise bending, chordwise bending, and torsion of a pretwisted nonuniform rotating blade are derived in Ref. 1. For simple harmonic free vibration with frequency ω, the governing equations are (see Fig. 1):

$$
\begin{aligned}
&-\{[GJ+Tk_A^2+EB_1(\beta')^2]\phi'-EB_2\beta'(v''\cos\beta+w''\sin\beta)\}'\\
&\quad+Te_A(v''\sin\beta-w''\cos\beta)+\Omega^2mxe(-v'\sin\beta+w'\cos\beta)\\
&\quad+\Omega^2me\sin\beta v+\Omega^2m[(k_{m_2}^2-k_{m_1}^2)\cos2\beta+ee_0\cos\beta]\phi\\
&\quad-\omega^2mk_m^2\phi+\omega^2me(v\sin\beta-w\cos\beta)=0
\end{aligned}
\tag{1a}
$$

$$
\begin{aligned}
&[(EI_1\cos^2\beta+EI_2\sin^2\beta)w''+(EI_2-EI_1)\sin\beta\cos\beta v''\\
&\quad-Te_A\phi\cos\beta-EB_2\beta'\phi'\sin\beta]''-(Tw')'\\
&\quad-(\Omega^2mxe\phi\cos\beta)'-\omega^2m(w+e\phi\cos\beta)=0
\end{aligned}
\tag{1b}
$$

$$
\begin{aligned}
&[(EI_2-EI_1)\sin\beta\cos\beta w''+(EI_1\sin^2\beta+EI_2\cos^2\beta)v''\\
&\quad+Te_A\phi\sin\beta-EB_2\beta'\phi'\cos\beta]''-(Tv')'+(\Omega^2mxe\phi\sin\beta)'\\
&\quad+\Omega^2me\phi\sin\beta-\Omega^2mv-\omega^2m(v-e\phi\sin\beta)=0
\end{aligned}
\tag{1c}
$$

where

$$T(x)=\int_x^R\Omega^2mx\mathrm{d}x \tag{2a}$$

$$B_1=\int_{\eta_{te}}^{\eta_{le}}t\eta^2(\eta^2+t^2/6-k_A^2)\,\mathrm{d}\eta \tag{2b}$$

$$B_2=\int_{\eta_{te}}^{\eta_{le}}t\eta(\eta^2+t^2/12-k_A^2)\,\mathrm{d}\eta \tag{2c}$$

and the notation is defined in the Nomenclature.

The derivation is made along the principles of engineering beam theory, and secondary effects, such as shear deformation and rotary inertia, are not included. The elastic, mass, and tension axes of the blade are assumed not to be coincident, and the coupling terms associated with the centrifugal forces are included.

The moment-displacement relations are given by:

$$
\begin{aligned}
Q&=[GJ+Tk_A^2+EB_1(\beta')^2]\phi'\\
&\quad-EB_2\beta'(v''\cos\beta+w''\sin\beta)
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
M_y&=(EI_1\cos^2\beta+EI_2\sin^2\beta)w''+(EI_2-EI_1)\sin\beta\cos\beta v''\\
&\quad-EB_2\beta'\phi'\sin\beta-Te_A\cos\beta\phi
\end{aligned}
\tag{3b}
$$

$$
\begin{aligned}
M_z&=(EI_2-EI_1)\sin\beta\cos\beta w''+(EI_1\sin^2\beta+EI_2\cos^2\beta)v''\\
&\quad-EB_2\beta'\phi'\cos\beta+Te_A\sin\beta\phi
\end{aligned}
\tag{3c}
$$

The relations between the shear forces and the bending moments are as follows:

$$V_z=-M_y'+Tw'+\Omega^2mxe\cos\beta\phi \tag{4a}$$

$$V_y=-M_z'+Tv'-\Omega^2mxe\sin\beta\phi \tag{4b}$$

The elastic properties and mass distribution may vary continuously or discontinuously along the length of the blade. The moments Q, M_y, and M_z and the shear forces V_z and V_y are required to be continuous at all locations.

The rotor blades for practical use either have fixed-free ends or hinged-free ends. The boundary conditions, therefore, are given by

$$\begin{aligned} &Q=M_y=M_z=V_y=V_z=0 && \text{for free end at } x=R \\ &\phi=w=v=w'=v'=0 && \text{for fixed end at } x=0 \\ &\phi=w=v=M_y=M_z=0 && \text{for hinged end in both the } y \text{ and } z \text{ directions at } x=0 \end{aligned} \tag{5}$$

III. New Quotient Approach

In this section, the general form of the new quotient is formulated. Redefine the bending moment and displacement relations in Eqs. (3b) and (3c) as

$$\mathring{M}_y=(EI_1\cos^2\beta+EI_2\sin^2\beta)w''+(EI_2-EI_1)\sin\beta\cos\beta v'' - EB_2\beta'\sin\beta\phi' \tag{6a}$$

$$\mathring{M}_z=(EI_2-EI_1)\sin\beta\cos\beta w''+(EI_1\sin^2\beta+EI_2\cos^2\beta)v'' - EB_2\beta'\cos\beta\phi' \tag{6b}$$

Note that $\mathring{M}_y$ and $\mathring{M}_z$ satisfy the same boundary conditions as M_y and M_z. This can be seen by examining the additional terms in the expressions of M_y and M_z, i.e., $Te_A\cos\beta\phi$ and $Te_A\sin\beta\phi$. For the free end at $x=R$, the tension force T (given by Eq. (2a)) vanishes. For the hinged end at $x=0$, ϕ is equal to zero. Hence $\mathring{M}_y$ and $\mathring{M}_z$ are zero at the hinged and free ends. The boundary conditions may, therefore, be written as

$$\begin{aligned} &Q=\mathring{M}_y=\mathring{M}_z=V_y=V_z=0 && \text{for free end at } x=R \\ &\phi=w=v=w'=v' && \text{for fixed end at } x=0 \\ &\phi=w=v=\mathring{M}_y=\mathring{M}_z=0 && \text{for a hinged end in both the } y \text{ and } z \text{ directions at } x=0 \end{aligned} \tag{7}$$

Substituting Eqs. (3a) and (6) into Eq. (1), we arrive at

$$\begin{aligned} &Q'+Te_A(v''\sin\beta-w''\cos\beta)+\Omega^2mxe(-v'\sin\beta+w'\cos\beta) \\ &+\Omega^2me\sin\beta v+\Omega^2m[(k_{m_2}^2-k_{m_1}^2)\cos2\beta+ee_0\cos\beta]\phi \\ &-\omega^2m[k_m^2\phi+e(\cos\beta w-\sin\beta v)]=0 \end{aligned} \tag{8a}$$

$$\mathring{M}_y''-(Te_A\cos\beta\phi)''-(Tw')'-(\Omega^2mxe\cos\beta\phi)' -\omega^2m(w+e\cos\beta\phi)=0 \tag{8b}$$

$$\mathring{M}_z''+(Te_A\sin\beta\phi)''-(Tv')'+(\Omega^2mxe\sin\beta\phi)' +\Omega^2me\sin\beta\phi-\Omega^2mv-\omega^2m(v-e\sin\beta\phi)=0 \tag{8c}$$

From Eqs. (3a) and (6) we express ϕ', w'', and v'' in terms of Q, $\mathring{M}_y$, and $\mathring{M}_z$, and obtain the following equations:

$$\phi'-E^2I_1I_2Q/\Delta-E^2I_1B_2\beta'\sin\beta\mathring{M}_y/\Delta -E^2I_1B_2\beta'\cos\beta\mathring{M}_z/\Delta=0 \tag{9a}$$

$$\begin{aligned} &w''-E^2I_1B_2\beta'\sin\beta Q/\Delta-\{[GJ+Tk_A^2+EB_1(\beta')^2] \\ &\times[EI_1\sin^2\beta+EI_2\cos^2\beta]-E^2B_2^2(\beta')^2\cos^2\beta\}\mathring{M}_y/\Delta \\ &-\{E^2B_2^2(\beta')^2\sin\beta\cos\beta-[GJ+Tk_A^2+EB_1(\beta')^2] \\ &\times(EI_2-EI_1)\sin\beta\cos\beta\}\mathring{M}_z/\Delta=0 \end{aligned} \tag{9b}$$

$$\begin{aligned} &v''-E^2I_1B_2\beta'\cos\beta Q/\Delta-\{E^2B_2^2(\beta')^2\sin\beta\cos\beta \\ &-[GJ+Tk_A^2+EB_1(\beta')^2](EI_2-EI_1)\text{s}\sin\beta\cos\beta\}\mathring{M}_y/\Delta \\ &-\{[GJ+Tk_A^2+EB_1(\beta')^2][EI_1\cos^2\beta+EI_2\sin^2\beta] \\ &-E^2B_2^2(\beta')^2\sin^2\beta\}\mathring{M}_z/\Delta=0 \end{aligned} \tag{9c}$$

where

$$\Delta=[GJ+Tk_A^2+EB_1(\beta')^2]E^2I_1I_2-E^3I_1B_2^2(\beta')^2 \tag{10}$$

Equations (8) and (9) with the boundary conditions, Eqs. (7), are the Euler-Lagrange equations of the following quotient:

$$\omega_N^2=\Delta_1/\Delta_2 \tag{11}$$

where

$$\begin{aligned} \Delta_1=\;&2\langle\mathring{M}_z,v''\rangle+2\langle Te_A\sin\beta\phi,v''\rangle+\langle Tv',v'\rangle \\ &-2\langle\Omega^2mxe\sin\beta\phi,v'\rangle+2\langle\Omega^2me\sin\beta\phi,v\rangle \\ &-\langle\Omega^2mv,v\rangle+2\langle\mathring{M}_y,w''\rangle-2\langle Te_A\cos\beta\phi,w''\rangle \\ &+\langle Tw',w'\rangle-2\langle\Omega^2mxe\cos\beta\phi,w'\rangle-2\langle Q,\phi'\rangle \\ &+\langle\Omega^2m[(k_{m_2}^2-k_{m_1}^2)\cos2\beta+ee_0\cos\beta]\phi,\phi\rangle \\ &+(1/\Delta)\{\langle E^2I_1I_2Q,Q\rangle+2\langle E^2I_1B_2\beta'\sin\beta\mathring{M}_y,Q\rangle \\ &+\langle E^2I_1B_2\beta'\cos\beta\mathring{M}_z,Q\rangle+\langle\{[GJ+Tk_A^2+EB_1(\beta')^2] \\ &[EI_1\sin^2\beta+EI_2\cos^2\beta]-E^2B_2^2(\beta')^2\cos\beta\}\mathring{M}_y,\mathring{M}_y\rangle \\ &+\langle\{[GJ+Tk_A^2+EB_1(\beta')][EI_1\cos^2\beta+EI_2\sin^2\beta] \\ &-E^2B_2^2(\beta')^2\sin^2\beta\}\mathring{M}_z,\mathring{M}_z\rangle \\ &+2\langle\{E^2B_2^2(\beta')^2\sin\beta\cos\beta-[GJ+Tk_A^2+EB_1(\beta')^2] \\ &(EI_2-EI_1)\sin\beta\cos\beta\}\mathring{M}_z,\mathring{M}_y\rangle\} \end{aligned}$$

$$\Delta_z=\langle mk_m^2\phi,\phi\rangle+\langle mw,w\rangle+\langle mv,v\rangle +2\langle me\cos\beta w,\phi\rangle-2\langle me\sin\beta v,\phi\rangle$$

and where the inner product is defined as:

$$\langle\alpha g,h\rangle\equiv\int_0^R\alpha gh\,\mathrm{d}x \tag{12}$$

Δ is defined in Eq. (10), and $\phi,w,v,Q,\mathring{M}_y,\mathring{M}_z$ are the six independent fields. ω_N^2 is called the "new quotient."

To obtain estimates for the eigenfrequencies and the corresponding eigenmodes, the Rayleigh-Ritz procedure will be used. To this end we set

$$\begin{aligned} &\bar{\phi}=\sum_{n=1}^N a_n\phi_n(x), && \bar{Q}=\sum_{n=1}^N b_nQ_n(x) \\ &\bar{w}=\sum_{n=1}^N c_nw_n(x), && \bar{\mathring{M}}_y=\sum_{n=1}^N d_nM_{yn}(x) \\ &\bar{v}=\sum_{n=1}^N e_nv_n(x), && \bar{\mathring{M}}_z=\sum_{n=1}^N f_nM_{zn}(x) \end{aligned} \tag{13}$$

where $a_n, b_n, c_n, d_n, e_n, f_n$ are unknown coefficients, and $\phi_n, Q_n, w_n, M_{yn}, v_n, M_{zn}$ are the normal modes of vibration of uniform beams which satisfy the fixed-free or hinged-free boundary conditions (cf. Refs. 18 and 22). Note that the condition of zero shear force at the free end and the condition of continuity of the shear force are considered as "natural conditions" (cf. Refs. 21 and 23); they are relaxed when beam functions are used in the Ritz procedure. For the particular case of $e=e_A$, these conditions are satisfied exactly as can be deduced from Eqs. (3), (6), and (4) by appropriate calculations.

Substituting Eq. (13) into Eq. (11), and setting equal to zero the derivatives of ω_N^2 with respect to the unknown coefficients a_n, b_n, c_n, d_n, e_n, and f_n, we arrive at a set of linear homogeneous matrix equations for estimating the eigenvalue ω^2. The solution of this matrix eigenvalue problem yields the estimates for the first $3N$ natural frequencies, denoted by $\omega_N^{(p)}$, $p=1,2,\ldots,3N$, and the corresponding eigenvectors $\{a_n^{(p)}\}$, $\{b_n^{(p)}\}$, $\{c_n^{(p)}\}$, $\{d_n^{(p)}\}$, $\{e_n^{(p)}\}$, and $\{f_n^{(p)}\}$. Substituting these eigenvectors into Eq. (13), we obtain the corresponding eigenmodes for the displacements and moments. Note that the moment mode shapes obtained by the preceding approach are continuous, regardless of the discontinuities in the blade elastic properties.

IV. Illustrative Examples

Two examples will be given here to demonstrate the application of the method of the new quotient.

Case 1: Pure Flapwise Bending, $e=e_A=\beta=0$; $v=\phi=0$

The blade considered has discontinuous mass and stiffness distributions which are given by Ref. 8

$$EI_1(x)\ddagger=\begin{cases}2.5\times 10^7 \text{ lb in.}^2, & 0\le x\le 0.2R\\ (4.332005-15.366799x+26.596032x^2 & \\ 15.153439x^3)\times 10^7 \text{ lb in.}^2, & 0.2\%<x\le R\end{cases}$$

$$m(x)\S=\begin{cases}0.397549+93.7898x-462.665x^2 \text{ lb}_m/\text{in.}, & \\ 0\le x\le 0.2R \quad 1.101767-0.512333x \text{ lb}_m/\text{in.}, & \\ 0.2R<x\le R\end{cases}$$

In the metric system, these become

$$EI_1(x)=\begin{cases}7.174175\times 10^4 \text{ Nm}^2, & 0\le x\le 0.2R\\ (15.454032-44.097642x+73.452165x^2 & \\ 43.485369x^3)\times 10^4 \text{ Nm}^2, & 0.2R<x\le R\end{cases}$$

$$m(x)=\begin{cases}7.09943+1674.8982x-8262.2716x^2 \text{ kg/m}, & \\ 0\le x\le 0.2R \quad 19.6675355-9.149243x \text{ kg/m}, & \\ 0.2R<x\le R\end{cases}$$

where $R=11.0$ m (432 in.). The blade is rotating at angular velocity, $\Omega=20$ rad/s, and is hinged at the root and is free at the other end.

The new quotient for this case is given by

$$\omega_N^2=\{2\langle M_y,w''\rangle+\langle Tw',w'\rangle-(1/EI_1)\langle M_y,M_y\rangle\}/\langle mw,w\rangle \tag{14}$$

In Eq. (14), M_y is used instead of $\mathring{M}_y$, since they are equivalent here.

Because of the hinged-free end conditions, the first mode is a rigid body rotation with frequency equal to Ω.[24] This mode must be included while applying the Rayleigh-Ritz procedure.

‡The conversion factor to N m^2 is 0.00286967.
§The conversion factor to kg/m is 17.8580.

The approximating functions for w and M_y are assumed to be:

$$\bar{w}=\sum_{n=1}^{N} c_n w_n(x)+c_0 w_0(x) \tag{15a}$$

$$\bar{M}_y=\sum_{n=1}^{N} d_n M_{yn}(x) \tag{15b}$$

where

$$w_n(x)=\sin\alpha_n x+A_n sh\alpha_n x$$

$$w_0(x)=x \text{ (rigid body rotation mode)}$$

$$M_{yn}(x)=\sin\alpha_n x-A_n sh\alpha_n x$$

$$A_n=\cos\alpha_n/ch\alpha_n$$

$$\alpha_1=3.9266,\quad \alpha_2=7.0686,\quad \alpha_3=10.2102,$$

$$\alpha_4=13.352,\quad \alpha_5=16.493,\quad \alpha_n=(n+1/4)\pi$$

$$\text{for } n\ge 6 \tag{16}$$

The values of α_n are given in Ref. 22. w_n and M_{yn} satisfy the hinged-free boundary conditions.

Substituting Eq. (15) into Eq. (14), and extremizing with respect to the unknown coefficients c_n and d_n, we obtain

$$\langle\bar{M}_y,w_n''\rangle+\langle T\bar{w}',w_n'\rangle-\omega_N^2\langle m\bar{w},w_n\rangle=0,\quad n=0,1,2,\ldots,N \tag{17a}$$

$$\left\langle\frac{1}{EI_1}\bar{M}_y,M_{yn}\right\rangle-\langle\bar{w}'',M_{yn}\rangle=0,\quad n=1,2,\ldots,N \tag{17b}$$

These equations can be rewritten in the form

$$\boldsymbol{Hd}+\boldsymbol{J}^\circ\boldsymbol{c}^\circ-\omega_N^2\boldsymbol{K}^\circ\boldsymbol{c}^\circ=0 \tag{18a}$$

$$\boldsymbol{Gd}-\boldsymbol{Hc}=0 \tag{18b}$$

where $\boldsymbol{c}$ and $\boldsymbol{d}$ are N-dimensional vectors, $\boldsymbol{c}^\circ$ is an $(N+1)$-dimensional vector with components $c_0, c_1, c_2, \ldots, c_N$, and the $N\times N$ matrices $\boldsymbol{G}$, $\boldsymbol{H}$, and the $(N+1)\times(N+1)$ matrices $\boldsymbol{J}^\circ$, $\boldsymbol{K}^\circ$ have the following components:

$$G_{np}=\langle(1/EI_1)M_{yn},M_{yp}\rangle \tag{19a}$$

$$H_{np}=\langle w_n'',M_{yp}\rangle=\langle w_p'',M_{yn}\rangle\quad n,p=1,2,\ldots,N \tag{19b}$$

$$J_{np}^\circ=\langle Tw_n',w_p'\rangle \tag{19c}$$

$$K_{np}^\circ=\langle mw_n,w_p\rangle\quad n,p=0,1,2,\ldots,N \tag{19d}$$

In Eq. (19c), the tension $T(x)$ is computed from Eq. (2a).

Equation (18) can be solved in the following way: From Eq. (18b), we obtain

$$\boldsymbol{d}=\boldsymbol{G}^{-1}\boldsymbol{Hc} \tag{20}$$

and substituting Eq. (20) into Eq. (18a), we have

$$\boldsymbol{Lc}+\boldsymbol{J}^\circ\boldsymbol{c}^\circ-\omega_N^2\boldsymbol{K}^\circ\boldsymbol{c}^\circ=0 \tag{21}$$

where $\boldsymbol{L}=\boldsymbol{HG}^{-1}\boldsymbol{H}$.

Equation (21) is equivalent to

$$\boldsymbol{L}^\circ\boldsymbol{c}^\circ+\boldsymbol{J}^\circ\boldsymbol{c}^\circ-\omega_N^2\boldsymbol{K}^\circ\boldsymbol{c}^\circ=0 \tag{22}$$

where

$$[L^\circ_{np}] = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & L_{11} & L_{12} & \dots & L_{1N} \\ 0 & L_{21} & L_{22} & \dots & L_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & L_{N1} & L_{N2} & \dots & L_{NN} \end{bmatrix}$$

Equation (22) defines an $(N+1)\times(N+1)$ matrix eigenvalue problem. The solution of Eq. (22) gives the first $(N+1)$ approximate eigenfrequencies $\omega_N^{(p)}$, $p=0,1,2,\dots,N+1$, and the corresponding eigenvector $c^{\circ(p)}$. The approximate eigenmodes for the displacement and the bending moment are obtained in the following manner. From Eq. (15a), we have

$$\bar{w}^{(p)} = \sum_{n=0}^{N} c_n^{(p)} w_n(x) \quad p=0,1,2,\dots,N \tag{23}$$

and from Eqs. (15b) and (20), we obtain

$$\bar{M}_y^{(p)} = \sum_{n=1}^{N} d_n^{(p)} M_{yn}(x) \quad p=1,2,\dots,N \tag{24}$$

Note that the moment mode corresponding to the rigid body motion is zero, and is not included in Eq. (24).

Case 2: Combined Flapwise Bending and Chordwise Bending with Pretwisted Deformation, $e=e_A=B_2=0$; $\phi=0$

The blade selected for the analysis in this case is from Ref. 3. It has a length of 0.4572 m (18 in.) and is cantilevered at 0.1524 m (6 in.) from the axis of rotation. The nonuniform properties of the blade are described in the form of experimental data points, as given in Table 1. These properties are approximated by step functions in the new quotient approach with eight points of discontinuity. In each interval, the mean value is taken for the function value.

The new quotient for this case is given by:

$$\begin{aligned}\omega_N^2 = \{ & 2\langle M_y, w''\rangle + 2\langle M_z, v''\rangle + \langle Tw', w'\rangle \\ & + \langle Tv', v'\rangle - \Omega^2\langle mv, v\rangle \\ & - \langle(\sin^2\beta/EI_2 + \cos^2\beta/EI_1) M_y, M_y\rangle \\ & - \langle(\cos^2\beta/EI_2 + \sin^2\beta/EI_1) M_z, M_z\rangle \\ & - 2\langle(1/EI_2 - 1/EI_1)\sin\beta\cos\beta M_y, M_z\rangle\}/\{\langle mw, w\rangle \\ & + \langle mv, v\rangle\}\end{aligned} \tag{25}$$

In Eq. (25), M_y and M_z are used instead of $\mathring{M}_y$ and $\mathring{M}_z$ since they have the same forms.

Table 1 Physical properties of propeller blade in case 2: $x_0 = 6$ in., $R = 18$ in. (from Ref. 3)

x, in.	m, lb-s^2/in.^{2}	EI_1 lb-in.^{2}	EI_2 lb-in.^{2}	β, deg
0.0	1.026×10^{-3}	0.200×10^6	63×10^6	30.5
2.0	0.696	0.110	49	25.2
4.0	0.660	0.083	46	20.0
6.0	0.608	0.058	44	14.8
8.0	0.564	0.042	54	9.6
10.0	0.535	0.031	43	4.7
12.0	0.520	0.027	44	0
14.0	0.506	0.026	47	−4.2
16.0	0.498	0.025	51	−7.6
18.0	0.498	0.024	56	−10.0

For approximate solutions we set

$$\bar{w} = \sum_{n=1}^{N} c_n w_n(x), \quad \bar{v} = \sum_{n=1}^{N} e_n v_n(x)$$

$$\bar{M}_y = \sum_{n=1}^{N} d_n M_{yn}(x), \quad \bar{M}_z = \sum_{n=1}^{N} f_n M_{zn}(x) \tag{26}$$

where

$$\begin{aligned} & w_n(x) = \sin\alpha_n x - sh\alpha_n x + A_n(\cos\alpha_n x - ch\alpha_n x) = v_n(x) \\ & M_{yn}(x) = \sin\alpha_n x + sh\alpha_n x + A_n(\cos\alpha_n x + ch\alpha_n x) = M_{zn}(x) \\ & A_n = -(\sin\alpha_n + sh\alpha_n)/(\cos\alpha_n + ch\alpha_n) \\ & \alpha_1 = 1.8751, \quad \alpha_2 = 4.6941, \quad \alpha_3 = 7.8548 \\ & \alpha_4 = 10.9955, \quad \alpha_5 = 14.1372, \quad \alpha_n = (n-\tfrac{1}{2})\pi \text{ for } n\geq 6 \end{aligned} \tag{27}$$

The values of α_n have been given in Ref. 22. The trial functions w_n, v_n, M_{yn}, and M_{zn} satisfy the cantilever boundary conditions.

Following the Ritz procedure as described previously, and expressing the corresponding simultaneous equations in the matrix form, we finally arrive at

$$\begin{aligned} & \boldsymbol{H_1 d + J_1 c - \omega_N^2 K_1 c = 0} \\ & \boldsymbol{H_2 f + J_2 e - \Omega^2 K_2 e - \omega_N^2 K_2 e = 0} \\ & \boldsymbol{G_1 d + G_3 f - H_1 c = 0} \\ & \boldsymbol{G_3 d + G_2 f - H_2 e = 0} \end{aligned} \tag{28}$$

where $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, and $\boldsymbol{f}$ are N-dimensional vectors with components c_n, d_n, e_n, f_n, $n=1,2,\dots,N$, respectively, and the components of the $N\times N$ matrices $\boldsymbol{H_1}$, $\boldsymbol{H_2}$, $\boldsymbol{J_1}$, $\boldsymbol{J_2}$, $\boldsymbol{K_1}$, $\boldsymbol{K_2}$, $\boldsymbol{G_1}$, $\boldsymbol{G_2}$, and $\boldsymbol{G_3}$ are defined as follows:

$$\begin{aligned} & H_{1np} = \langle w_n'', M_{yp}\rangle = \langle w_p'', M_{yn}\rangle = H_{2np} \\ & J_{1np} = \langle Tw_n', w_p'\rangle = J_{2np} \\ & K_{1np} = \langle mw_n, w_p\rangle = K_{2np} \\ & G_{1np} = \langle(\sin^2\beta/EI_2 + \cos^2\beta/EI_1) M_{yn}, M_{yn}\rangle \\ & G_{2np} = \langle(\cos^2\beta/EI_2 + \sin^2\beta/EI_1) M_{zn}, M_{zn}\rangle \\ & G_{3np} = \langle(1/EI_2 - 1/EI_1)\sin\beta\cos\beta M_{yn}, M_{yn}\rangle \end{aligned} \tag{29}$$

As the root of the blade is at a distance of $x_0 = 0.1524$ m (6 in.) from the center of rotation, the tension T, in this case, is given by:

$$T(x) = \int_x^R \Omega^2 m(x+x_0)\,dx \tag{30}$$

To solve the system of matrix equations, Eq. (28), we rewrite it as

$$\boldsymbol{Hs + Jr - \omega_N^2 Kr = 0} \tag{31a}$$

$$\boldsymbol{Gs - Hr = 0} \tag{31b}$$

where s and r are $2N$-dimensional vectors defined by

$$s = \begin{bmatrix} d \\ f \end{bmatrix} \quad r = \begin{bmatrix} c \\ e \end{bmatrix} \tag{32}$$

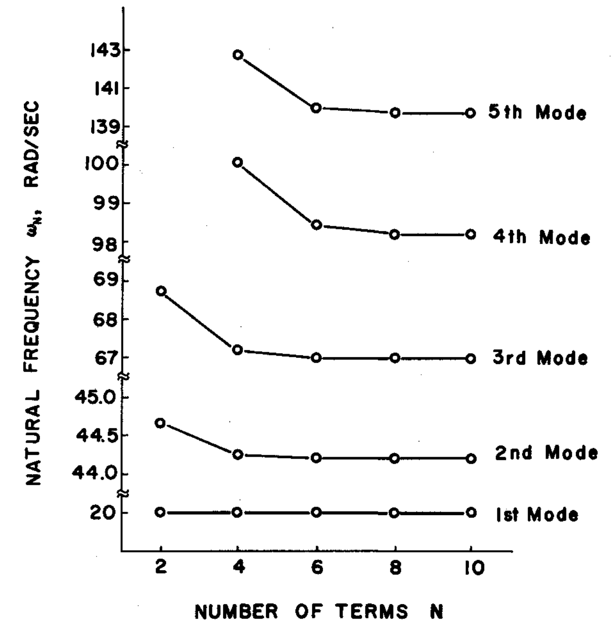

Fig. 2 Convergence tendency of the estimated frequencies: case 1.

Table 2 Natural frequencies: case 1, $N=9$

Mode	ω, rad/s		
	ω_N	Ref. 8	Ref. 7
1	20.00	—	20.00
2	44.19	44.19	44.19
3	67.01	67.01	67.01
4	98.18	98.16	98.09
5	139.74	139.67	139.55
6	186.88	—	—
7	237.38	—	—
8	290.33	—	—

Table 3 Mode shapes (displacement): case 1, $N=9$

x/R	2nd mode $\bar{w}$, Eq. (23)	2nd mode Ref. 8	3rd mode $\bar{w}$, Eq. (23)	3rd mode Ref. 8
0.0	0.00	0.00	0.00	0.00
0.1	−0.21	−0.21	0.28	0.28
0.2	−0.35	−0.35	0.32	0.32
0.3	−0.41	−0.41	0.17	0.17
0.4	−0.42	−0.42	−0.06	−0.06
0.5	−0.36	−0.36	−0.30	−0.30
0.6	−0.24	−0.23	−0.46	−0.46
0.7	−0.03	−0.03	−0.46	−0.46
0.8	0.25	0.25	−0.21	−0.21
0.9	0.61	0.61	0.30	0.30
1.0	1.00	1.00	1.00	1.00

x/R	4th mode $\bar{w}$, Eq. (23)	4th mode Ref. 8	5th mode $\bar{w}$, Eq. (23)	5th mode Ref. 8
0.0	0.00	0.00	0.00	0.00
0.1	−0.21	−0.21	0.19	0.19
0.2	−0.06	−0.06	−0.16	−0.16
0.3	0.26	0.26	−0.49	−0.48
0.4	0.47	0.46	−0.31	−0.30
0.5	0.39	0.38	0.26	0.25
0.6	0.02	0.02	0.55	0.54
0.7	−0.41	−0.41	0.14	0.14
0.8	−0.53	−0.53	−0.52	−0.50
0.9	−0.02	−0.03	−0.32	−0.31
1.0	1.00	1.00	1.00	1.00

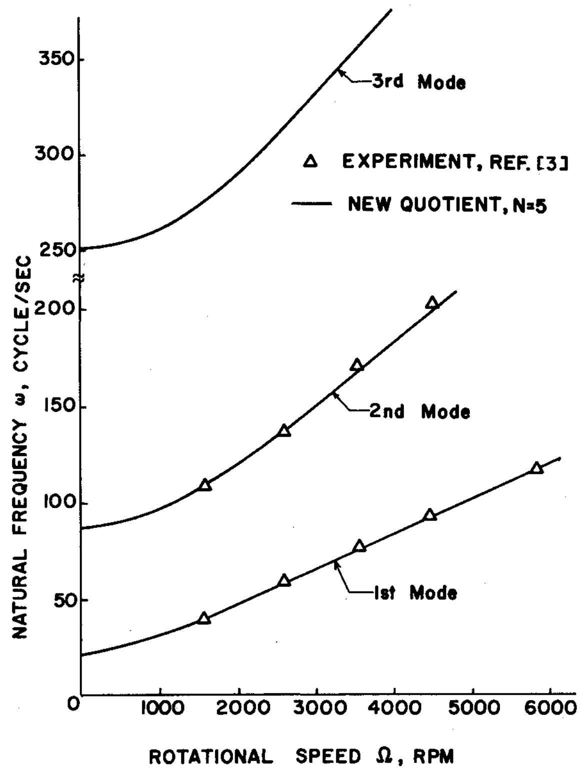

Fig. 3 Effect of rotational speed on the natural frequencies: case 2.

and the $2N \times 2N$ matrices H, J, K, and G are defined as:

$$H=\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \qquad J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2-\Omega^2 K_2 \end{bmatrix}$$

$$K=\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \qquad G=\begin{bmatrix} G_1 & G_3 \\ G_3 & G_2 \end{bmatrix} \tag{33}$$

Equations (31) can be solved similarly to the previous example. The obtained frequencies $\omega_N^{(p)}$, $p=1,2,...,2N$, are the estimates for the first $2N$ exact eigenfrequencies. The corresponding eigenmodes for the displacements and the moments are denoted by $\bar{w}^{(p)}$ and $\bar{v}^{(p)}$, and $\bar{M}_y^{(p)}$ and $\bar{M}_z^{(p)}$, respectively.

V. Discussion of Results

Table 2 presents the natural vibration frequencies for case 1, the hinged-free blade. The results obtained by the new quotient method show very good agreement with those obtained by the Runge-Kutta method,[8] and the transfer matrix method.[7] The numerical convergence of the new quotient approach is shown in Fig. 2. As can be seen, the natural frequencies converge rapidly to steady values. The eigenmode shapes for the displacement and the bending moment are given in Tables 3 and 4, and are compared with the Runge-Kutta solution.[8] For the sake of convenience in comparison, the displacement at the tip of the blade and the moment in the middle of the blade are normalized to 1. Note that the discontinuity of the blade elastic property at $x=0.2\,R$ does not affect the continuous distribution of the moment modes.

Table 5 gives the first three vibration frequencies of the fixed-free blade in case 2, where various rotational speeds are assigned. The dashes in this table mean that the numerical data are not reported in the corresponding references. Reference 3 gives the experimental results and the integrating matrix solutions, and Ref. 6 reports the results of the transfer matrix solution. As can be seen, the results obtained by the new quotient approach estimate the measured frequencies very accurately. The percent difference of the new quotient

Table 4 Mode shapes (moment): case 1, $N=9$

x/R	2nd mode $\bar{M}_y$, Eq. (24)	2nd mode Ref. 8	3rd mode $\bar{M}_y$, Eq. (24)	3rd mode Ref. 8
0.0	0.00	0.00	0.00	0.00
0.1	2.06	2.03	−5.55	−6.40
0.2	1.75	1.52	−3.85	−3.85
0.3	0.90	0.95	−0.91	−1.25
0.4	1.04	0.93	−0.24	−0.11
0.5	1.00	1.00	1.00	1.00
0.6	1.18	1.09	1.92	2.23
0.7	1.13	1.12	3.06	3.33
0.8	1.03	0.95	3.10	3.52
0.9	0.46	0.47	1.80	1.95
1.0	0.00	0.00	0.00	0.00

x/R	4th mode $\bar{M}_y$, Eq. (24)	4th mode Ref. 8	5th mode $\bar{M}_y$, Eq. (24)	5th mode Ref. 8
0.0	0.00	0.00	0.00	0.00
0.1	2.38	2.57	−4.05	−4.50
0.2	0.97	0.86	0.25	0.37
0.3	−0.55	−0.50	2.30	2.39
0.4	−0.96	−1.03	1.52	1.54
0.5	−1.00	−1.00	−1.00	−1.00
0.6	−0.23	−0.27	−2.66	−2.79
0.7	1.00	1.05	−1.11	−1.11
0.8	2.04	2.08	2.99	3.04
0.9	1.38	1.46	3.31	3.51
1.0	0.00	0.00	0.00	0.00

Table 5 Natural frequencies: case 2, $N=3$

Ω, rpm	Mode	ω_N	ω, cycle/s Expt. 3	Ref. 3	Ref. 6
1567	1	39.89	40.08	40.77	40.96
	2	107.40	—	—	109.22
	3	276.32	—	—	279.79
1589	1	40.26	—	—	41.35
	2	107.93	107.53	109.05	109.77
	3	276.97	—	—	280.47
2609	1	58.05	58.73	59.85	60.07
	2	135.99	—	—	139.52
	3	313.98	—		309.40
2614	1	58.14	—	—	60.16
	2	136.14	137.02	139.03	139.68
	3	314.19	—	—	319.62
3583	1	75.30	76.52	78.08	78.34
	2	166.25	—	—	—
	3	357.70	—	—	—

Table 6 Deviation from experimental results in the first two estimated frequencies: case 2, $N=5$

Mode	Ω, rpm	ω_N	ω, cycle/s Expt. 3	Difference, %
1	1567	39.90	40.08	0.4
	2609	58.12	58.73	1.0
	3583	75.41	76.52	1.4
	4486	91.42	93.07	1.8
	5884	116.07	117.50	1.2
2	1589	108.18	107.53	0.6
	2614	136.53	137.02	0.4
	3585	166.64	170.60	2.3
	4537	196.41	202.53	3.0

solutions is given in Table 6. Figure 3 shows graphically the effect of the rotational speed on the natural frequency. The curves obtained from the new quotient solution are in good agreement with the experimental data.

VI. Conclusion

The method of the new quotient has been shown to be very effective for the analysis of vibration of blades with large variations (or discontinuous changes) in the blade properties.

The nonuniform blade properties can be approximated by step functions in the new quotient approach. This reduces the required effort in the integration procedure, yet yields accurate estimates for the eigenfrequencies and the corresponding mode shapes. In addition, this kind of approximation has the advantage that the same computer program can be used for blades with different nonuniform properties by simply changing the blade input data expressed as step functions.

There is one major disadvantage in the use of the new quotient approach for the problem of blade vibration; namely, that the new quotient method does not give upper or lower bounds for the natural frequencies, since it is not based on a minimum or maximum principle. For certain problems this shortcoming can be overcome by obtaining bounds for the frequencies; see, for example, Refs. 18, 21, and 25. We have not addressed this question in the present work.

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